

Weak approximation of an invariant measure and a low boundary of the entropy

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Abstract

For a measurable map T and a sequence of T -invariant probability measures μ_n that converges in some sense to a T -invariant probability measure μ , an estimate from below for the Kolmogorov–Sinai entropy of T with respect to μ is suggested in terms of the entropies of T with respect to μ_1, μ_2, \dots

In problems of Ergodic theory and Thermodynamic formalism it is sometimes necessary to estimate the entropy of a measure preserving map. If this map acts in a compact metric space and is expansive, one can use the fact that the entropy is semicontinuous from above on the space of invariant probability measures. Both conditions — the compactedness and expansiveness are essential in this context, and if at least one of them fails, one has to use other means. One such a mean is suggested in this note.

We will use standard notation, terminology and results from Entropy theory (see, e.g., [1] – [3]). Let T be an automorphism of a measurable space (X, \mathcal{F}) and μ_0 a T -invariant probability measure. For a finite or countable infinite partition η of (X, \mathcal{F}) , we write $B \in \eta$ and $B \subset \eta$ if the set B is an atom of η or a union of such atoms, respectively.

Theorem. *Assume that for a countable measurable partition ξ of (X, \mathcal{F}) , the entropy $H_{\mu_0}(\xi)$ is finite and that there exist a sequence of T -invariant probability measures μ_n and sequences of numbers $r_n \in \mathbb{N}$, $\varepsilon_n > 0$ such that*

$$\lim_{n \rightarrow \infty} r_n = \infty, \quad \lim_{n \rightarrow \infty} \varepsilon_n = 0, \quad \limsup_{n \rightarrow \infty} h_{\mu_n}(T) \geq h \geq 0, \quad (1)$$

$$\xi \text{ is a generator for } (T, \mu_n), \quad n \geq 0, \quad (2)$$

$$|\mu_0(A) - \mu_n(A)| \leq \varepsilon_n \mu_n(A) \text{ for all } A \in \bigvee_{i=0}^{r_n} T^{-i} \xi, \quad n \geq 0. \quad (3)$$

Then $h_{\mu_0}(T) \geq h$.

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We begin the proof of the Theorem with two simple lemmas.

Lemma 1. *Let $\mathbf{p} := (p_i)_{i \in \mathbb{N}}$, $\mathbf{q} := (q_i)_{i \in \mathbb{N}}$, where $p_i, q_i \geq 0$ for all i and $\sum_i p_i = \sum_i q_i = 1$. Let also $H(\mathbf{p}) := -\sum_{i \in \mathbb{N}} p_i \ln p_i$ (with $0 \ln 0 = 0$) and $H(\mathbf{q})$ be defined similarly. Assume that for some $c \in (0, 1/3)$,*

$$|p_i - q_i| \leq cq_i, \quad i = 1, 2, \dots \quad (4)$$

Then

$$H(\mathbf{p}) \leq (1 + c)H(\mathbf{q}) + c \ln 3.$$

Proof. Denote $\varphi(t) := -t \ln t$, $t \geq 0$. It is clear that (a) $\varphi(t)$ increases when $0 \leq t \leq e^{-1}$, (b) $\varphi(t) \leq 0$ when $t \geq 1$, (c) $-1 \leq \varphi'(t) \leq \ln 3$ when $(3e)^{-1} \leq t \leq 1$. Hence (see also (4))

$$\begin{aligned} H(\mathbf{p}) &= \sum_{i \in \mathbb{N}} \varphi(p_i) = \sum_{i: q_i \leq 1/2e} \varphi(p_i) + \sum_{i: q_i > 1/2e} \varphi(p_i) \leq \sum_{i: q_i \leq 1/2e} \varphi((1+c)q_i) \\ &\quad + \sum_{i: q_i > 1/2e} [\varphi(q_i) + |p_i - q_i| \ln 3] = \sum_{i: q_i \leq 1/2e} [q_i \varphi(1+c) + (1+c)\varphi(q_i)] \\ &\quad + \sum_{i: q_i > 1/2e} [\varphi(q_i) + |p_i - q_i| \ln 3] \leq (1+c) \sum_{i \in \mathbb{N}} \varphi(q_i) + c \ln 3 = (1+c)H(\mathbf{q}) + c \ln 3. \end{aligned}$$

□

Lemma 2. *If η is a countable measurable partition of the space (X, \mathcal{F}) and if, for probability measures μ and ν on (X, \mathcal{F}) , for every $A \in \eta$ and some $\varepsilon > 0$, we have $|\mu(A) - \nu(A)| \leq \varepsilon \nu(A)$, then the same is true for every $A \subset \eta$.*

The proof is evident and will be omitted.

We continue the proof of the Theorem. For all $k, l \in \mathbb{Z}$, $k \leq l$, we denote $(\xi^T)_{-l}^{-k} := \vee_{i=l}^k T^{-i} \xi$.

It is known [3] that if $H_{\mu_0}(\xi) < \infty$ and (2) holds for $n = 0$, then

$$h_{\mu_0}(T) = \lim_{n \rightarrow \infty} \frac{1}{n} H_{\mu_0}((\xi^T)_{-n}^{-1}),$$

and the sequence on the right hand side is non-increasing.

It is easy to verify that if $\varepsilon \leq 1/2$, then $|a - b| \leq \varepsilon b$, $a, b \geq 0$ imply that $|a - b| \leq 2\varepsilon a$. Therefore by (3) for all n such that $\varepsilon_n \leq 1/2$ and all $A \in \xi(-r_n, 0)$, we have

$$|\mu_0(A) - \mu_n(A)| \leq 2\varepsilon_n \mu_0(A). \quad (5)$$

For these n , we compare $H_{\mu_0}((\xi^T)_{-r_n}^{-1})$ and $H_{\mu_n}((\xi^T)_{-r_n}^{-1})$.

An arbitrary numbering of the atoms $A \in (\xi^T)_{-r_n}^{-1}$ yields a sequence A_1, A_2, \dots . Let $p_i := \mu_n(A_i)$, $q_i := \mu_0(A_i)$. By applying Lemma 1 for $c = 2\varepsilon_n$ (see (5)) we obtain

$$H_{\mu_n}((\xi^T)_{-r_n}^{-1}) \leq (1 + 2\varepsilon_n)H_{\mu_0}((\xi^T)_{-r_n}^{-1}) + 2\varepsilon_n \ln 3.$$

For all sufficiently large n , this implies that

$$\begin{aligned} h_{\mu_n}(T, \xi) &\leq \frac{1}{r_n} H_{\mu_n}((\xi^T)_{-r_n}^{-1}) \\ &\leq \frac{1}{r_n} (1 + 2\varepsilon_n) H_{\mu_0}((\xi^T)_{-r_n}^{-1}) + \frac{2}{r_n} \varepsilon_n \ln 3, \end{aligned}$$

or

$$h_{\mu_n}(T, \xi) \leq \frac{1}{r_n} (1 + 2\varepsilon_n) H_{\mu_0}((\xi^T)_{-r_n}^{-1}) + \frac{2}{r_n} \varepsilon_n \ln 3.$$

Therefore (see (1))

$$h \leq \limsup_{n \rightarrow \infty} h_{\mu_n}(T, \xi) \leq h_{\mu_0}(T).$$

The proof is completed.

Remark 1. Let us number in an arbitrary way the atoms of the partition ξ , and consider the finite partition ξ_m obtained from ξ by replacing all atoms of ξ with labels $\geq m$ by their union. In the statement of the Theorem, the conditions $\limsup_{n \rightarrow \infty} h_{\mu_n}(T) \geq h$ (see (1)) and $H_{\mu_0}(\xi) < \infty$ can be changed for the single (and weaker) condition

$$\limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} h_{\mu_n}(T, \xi_m) \geq h.$$

References

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